JOURNAL OF COMPUTATIONAL PHYSICS 28, 147–162 (1978)

Series Representation of the Eigenvalues of the Orr–Sommerfeld Equation

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Received July 6, 1977; revised November 16, 1977

A series representation of the relation that links the eigenvalues of the Orr–Sommerfeld equation is developed. This enables the complex frequency parameter to be expressed as a double series in terms of the Reynolds number and wavenumber, both of which are treated as complex variables. The complex coefficients arising in this series are determined by contour integration for the case of the eigenfunctions for a Blasius boundary layer profile. A nonlinear transformation is applied to the partial summations formed from the series in order to improve the convergence, and so to enable predictions of high accuracy to be made from only a few terms. Eigenvalues calculated by this technique are compared with those obtained directly from the Orr–Sommerfeld equation. The power of the technique is demonstrated by various graphical displays of the amplification contours for both temporal and spatial modes.

1. INTRODUCTION

At high Reynolds numbers weak disturbances arise in laminar boundary layers as a result of excitation by turbulent fluctuations in the free-stream flow, or by vibrations of the boundary. The resulting disturbances, which take the form of traveling waves can be unstable in such a way that the amplitude of the associated velocity fluctuations imposed on the steady laminar boundary layer flow increases as the waves propagate downstream. Initially the amplitudes of these fluctuations will be too small for their nonlinear stress components to be important as far as the development of the mean boundary layer flow is concerned, but eventually, after the disturbance has undergone sufficient amplification, the Reynolds stresses will become large enough to modify the mean flow. Higher harmonics are also generated and the complex nonlinear interactions that then take place, together with the randomly excited secondary instabilities, ultimately lead to turbulent flow.

This paper is concerned with predictions of the possible modes of instability that occur during the linear phase of the above process, and in particular with the calculation of the rates at which the various traveling wave disturbances amplify.

Even when the equations of motion that describe the development of these small periodic disturbances are linearized, they remain partial differential equations and their solution is by no means straightforward. At the relatively high Reynolds numbers

0021-9991/78/0292–0147$02.00/0
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at which instability occurs, the mean boundary layer flow changes in thickness relatively slowly in the downstream direction, and the approximation is therefore commonly made, that as far as the behavior of the perturbation is concerned, the boundary layer can be treated locally as a parallel flow. With this simplification the equation defining the disturbance reduces to an ordinary differential equation, which can be solved numerically without very much difficulty, although the operation is still time consuming. The equation is known as the Orr–Sommerfeld equation, and solutions of it that are compatible with the usual hydrodynamic boundary condition at the wall and in the free stream lead to a characteristic function defining the eigenvalues. The behavior of any particular mode propagating along the boundary layer is then given in terms of integrals of these local estimates of the wave length and amplification rate. Concern has often been expressed over the validity of this "quasi-parallel" approach to the prediction of the spatial development of a wave over large streamwise distances. On solving the full linearized partial differential equations by a multiple scale technique, where the unstable perturbations of a nearly parallel flow are expanded in terms of a suitable small parameter defining the deviation from the parallel, it is found that the zero-order term is precisely that of the locally parallel flow solution. A first-order correction to this can be provided by a scaling factor that is a function of the streamwise distance. Although it is found that this correction term lowers the critical Reynolds number slightly, its overall influence on the growth rate is small and calculations of amplification based on the zero-order analysis are sufficiently accurate for most practical cases. In the following treatment the locally parallel mean flow treatment will be used. Even with the foregoing simplifying approximations it turns out that predicting the evolution of even a single unstable traveling wave, defined in terms of its frequency and spanwise wavelength, can be quite a lengthy computation, because eigenvalues are required at a large number of streamwise locations. It is shown in this paper how series expansions of the eigenvalue relations can be used to reduce the effort in calculations that involve large numbers of eigenvalues. This technique is applied to the instability modes that arise in the Blasius boundary layer velocity profile.

A knowledge of the amplification rates of the various unstable modes can be useful when attempting to estimate the most likely position of transition to a turbulent flow. In fact, since most of the amplification takes place in the linear regime, reasonably valid predictions of the overall growth of a disturbance can be made from theories that neglect the later nonlinear behavior. Thus, although the final phase of the transition process is not at all well understood, estimates of the position where transition eventually occurs may be made on the basis of amplification factors computed from linear stability theory. Simple transition prediction rules like the "e^9" law rely on the idea that transition occurs when the overall growth of naturally occurring disturbances reaches a given level. The direct computation of overall growth factors for a range of wavenumbers and frequencies can be very expensive in computer time, and ways of reducing this are woth considering.

A rapid way of calculating eigenvalues can also be invaluable when making predictions of the evolution of more complex linear disturbances. For example, a simple wave packet initiated by an impulsive excitation at a point can be synthesized by a summation of traveling wave modes of all frequencies and spanwise wave numbers [1]. This type of calculation can also involve a lot of computational effort when the eigenvalues are calculated individually from the differential equation. The asymptotic description of a wave packet also requires values of various derivatives, like the group velocity, and again if these are obtained from the basic differential equation the processes take rather a long time [2]. The series method presented here lends itself particularly well to both of the above types of calculation.

2. Analysis

In a previous paper by Gaster and Jordinson [3] the series technique was used to define the eigenvalue relationships for the two-dimensional modes derived from the Orr–Sommerfeld equation for a Blasius profile at a fixed boundary layer Reynolds number of 1000. In that case the disturbance was represented by a stream function of the form $\phi(y, \alpha, \beta)$ expi $(\alpha x - \beta t)$ where y is normal to the boundary and x is in the stream direction. The differential equation for $\phi(y)$ is

$$(\alpha U(y) - \beta)(\phi'' - \alpha^2\phi) - \alpha U''(y)\phi = -\frac{i}{R}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi), \tag{1}$$

where the local mean flow velocity profile is given by $U(y)$. The wave number α and the frequency parameter β that arise in the above equation have eigenvalues defined by a characteristic function of the form $F(\alpha, \beta) = 0$. Both parameters may be considered to be complex variables; temporal modes are defined by those with β complex and α real, while spatial ones have real β and complex α. In the Blasius boundary layer flow problem there is only one unstable mode, and the ensuing work is concerned solely with the evaluation of this lowest mode. For this one discrete mode it was shown in [3] how $F(\alpha, \beta)$ can be expanded about an arbitrary point (α_0, β_0) as Taylor series in $(\alpha - \alpha_0)$ and $(\beta - \beta_0)$. It is then possible to express β as a series in $(\alpha - \alpha_0)$, or α as $(\beta - \beta_0)$, except in those regions where there are branch points. There must, of course, always be branch points through which links with higher modes occur. In those regions where an analytic behavior does exist, the coefficients of the power series can be found by appropriate contour integrations. The above formulation of the characteristic equation was used to provide a link between purely spatial and purely temporal modes.

Here we consider a more general problem of the eigenvalues of the Orr–Sommerfeld equation when the Reynolds number, R, is also treated as an additional complex parameter. The resulting characteristic function then becomes $F(\alpha, \beta, R)$, and as in [3], one can expect β to be a analytic function of both α and R in restricted regions of the α and R planes. It turns out to be more convenient to use the parameters that occur naturally in the basic equation, and to express β/α as a function of α^2 and αR. This formulation is especially convenient when extending the method to oblique waves.

3. The Expansion

In regions of the α^2–αR planes where there are no singularities, β/α will be an analytic function of these parameters and can be expressed as a sum of general solutions of the form

$$A_{nm} r_1^{\,n} e^{in\theta_1} r_2^{\,m} e^{im\theta_2}, \tag{2}$$

where $r_1 e^{i\theta_1} = (\alpha R - (\alpha R)_0)$, $r_2 e^{i\theta_2} = (\alpha^2 - \alpha_0^{\,2})$, and the subscript 0 denotes the values of a parameter at the origin of the expansion. A_{nm} is in general a complex coefficient. In [3] the coefficients of the power series defining β in terms of α were determined from the Fourier transform of a number of discrete evaluations of β at points equally spaced around a circle in the α plane. Provided a sufficient number of discrete points was used, the resulting finite series adequately represented the function within the circular regions. In this extension of [3] the same procedure has been adopted to obtain the A_{nm} coefficients arising in Eq. (2), but here circuits had to be made in both the α^2 and αR planes. N points were chosen equally spaced around a circular region in the α^2 plane, and at each location αR was also varied around a circular contour in M steps. At each of these N, M values of the parameters α^2 and αR the frequency parameter β was evaluated. The resulting double series contained A_{nm} coefficients for n up to $N/2$ and m to $M/2$, making a total of 400 in the present case.

4. Computations

A Blasius boundary layer profile defined by the differential equation $2f''' + ff'' = 0$ was used as the mean flow in the Orr-Sommerfeld equation, where $U/U_\infty = f'(\eta)$ and $\eta = y(V_\infty/\nu x)^{1/2}$ with the boundary conditions $f(0) = f'(0) = 0$ and $f'(y) \to 1$ for large η. U_∞ is the freestream velocity, x is the distance from the leading edge, and ν is the kinematic viscosity. For the subsequent solution of the Orr–Sommerfeld equation, $f'(\eta)$ and $f'''(\eta)$ were required at 640 equispaced intervals within the range $0 < \eta < 10$. A Runge–Kutta integration scheme was used to solve the differential equation for f with iteration on the value of $f''(0)$ until the outer boundary condition was satisfied.[1] The displacement thickness δ^*, was found to be

$$\delta^* = 1.72078766(x\nu/U_\infty)^{1/2}.$$

Eigenvalues of the Orr-Sommerfeld equation for the appropriate values of αR and α^2 were then evaluated by a shooting technique which used the previously stored values of f' and f at the 640 steps in the range η from 0 to 10. For specific values of

[1] Although the solution of this equation for arbitrary $f''(0)$ can be scaled with a "constant of homology" to fit the outer boundary behaviour, the integration range and step length cannot then easily be set to predetermined values.

β, α, and R, the two decaying solutions determined analytically in the free stream were numerically integrated towards the boundary wall by a Runge–Kutta scheme over 320 equispaced intervals. Single precision arithmetic (11 decimal digits on the KDF9) was used in conjunction with a filtering scheme designed to remove the spurious divergent mode that can allow rounding errors to build up and contaminate the integration. The determinant defining the boundary conditions was reduced to a very small value by modifying β through a Newton–Raphson iteration scheme. Values obtained from this procedure for three different values of α and R are compared below with values given by Davey which are accurate to 8 decimal places. The eigenvalues obtained by the present method appear to be accurate to 7 decimal places. The differences probably arise because fewer integration steps were used in the present exercise.

Case	α	R	Values of	β_r and β_i	Values given by Davey	
(a)	0.3	500	0.11930374	−0.00027997	0.11930376	−0.00027998
(b)	0.2	1500	0.06312288	0.00315665	0.06312291	0.00315663
(c)	0.15	3000	0.04021914	0.00278070	0.04021919	0.00278068

5. The Series

N and M were chosen to be equal to 40, and values of β were computed by the shooting technique for all 1,600 values of N, M, equispaced around circles in the α^2 and αR planes, where $(\alpha R)_0 = 460$; $(\alpha)_0^2 = 0.063$ and the two radii are $r_1 = 225$; $r_2 = 0.04275$. These values were used in the calculation of the 400 A_{nm} coefficients. Figure 1 shows how the modulus of these coefficients depends on both n and m. Since the modulus falls fairly rapidly with respect to both of these parameters, the series representation of an eigenvalue will be a good one when αR and α^2 lie within the regions defined above, and the closer these parameters are to $(\alpha R)_0$ and $(\alpha)_0^2$, the better the rate of convergence of the series and the more accurate the result. In order to examine the convergence of the sum of terms (2) it is convenient to define the qth partial sum of the double series as

$$S(q) = \sum_{n=0}^{q} \sum_{m=0}^{q} A_{nm}(\alpha R - (\alpha R)_0)^n(\alpha^2 - \alpha_0{}^2)^m. \tag{3}$$

For case (b), where $\alpha = 0.2$ and the Reynolds number is 1500, the sequence is given in Table I. The series has converged to a limit that is consistent to eight decimal places with the value obtained by direct solution of the Orr–Sommerfeld equation. Since the same numerical integration procedure was also used to calculate the N, M eigenvalues that formed the coefficients, this value can be considered to be "exact" as far as assessing the accuracy of the series.

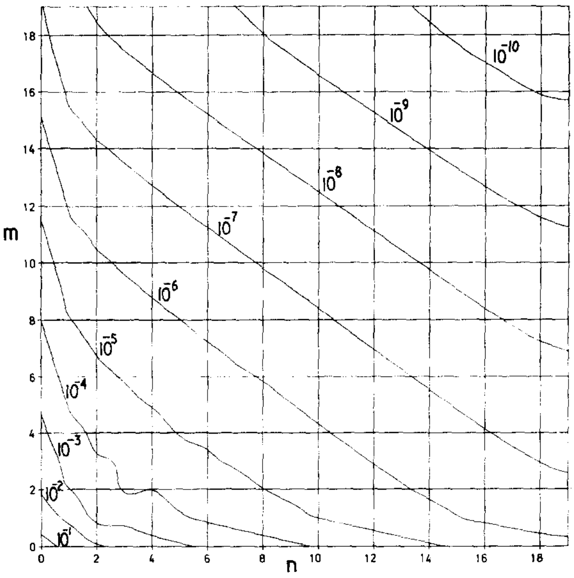


FIG. 1. The modulus of the Fourier coefficients A_{nm}.

TABLE I

q	$S_r(q)$	$S_i(q)$
0	0.06446508	0.00322617
1	0.06331076	0.00399932
2	0.06314974	0.00343594
3	0.06312942	0.00324496
4	0.06312509	0.00318464
5	0.06312373	0.00316559
6	0.06312321	0.00315953
7	0.06312301	0.00315758
8	0.06312293	0.00315695
9	0.06312289	0.00315675
10	0.06312288	0.00315668
11	0.06312288	0.00315666
12	0.06312288	0.00315665
Exact value	0.06312288	0.00315665

A more demanding test of the series solution occurs when the values of α^2 or αR lie towards the edge, or even outside, the original circular contour regions. Such an example is provided by case (a), where the radius r_1 is 1.216. Convergence is quite rapid with respect to m, but is slow as far as n is concerned, and it is convenient to define partial sums with respect to n as

$$S(q) = \sum_{n=0}^{q} \sum_{m=0}^{14} A_{nm}(\alpha R - (\alpha R)_0)^n(\alpha^2 - \alpha_0{}^2)^m. \tag{4}$$

TABLE II

w	$S_r(q)$	$S_i(q)$
0	0.10258149	0.00243754
1	0.11188303	0.00411893
2	0.11557909	0.00263507
3	0.11731007	0.00167125
4	0.11819797	0.00098586
5	0.11867527	0.00054261
6	0.11893995	0.00025283
7	0.11909018	0.00006551
8	0.11917686	−0.00005589
9	0.11922759	−0.00013448
10	0.11925762	−0.00018542
11	0.11927557	−0.00021847
12	0.11928640	−0.00023994
13	0.11929298	−0.00025389
14	0.11929702	−0.00026297
Exact value	0.11930374	−0.00027997

The series (Table II) seems to be converging towards the exact value, but the 14th partial sum is still only correct to four decimal places. Shanks [4] discusses a number of possible nonlinear operations that can be used to improve the rate of convegence of series. The simplest of these operations provides a new sum, $\hat{S}(q)$, from the relation

$$\hat{S}(q) = \frac{S(q-1)\,S(q+1) - S^2(q)}{S(q-1) + S(q+1) - 2S(q)}. \tag{5}$$

This transformation has been applied to the series in Table II, and the resulting values of the real and imaginary parts of the new series are tabulated in the first columns of Tables III and IV, respectively. The new series in column (i) is certainly more convergent than the original one, and it seems sensible therefore to repeat the process

TABLE III

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
0.11629130						
0.11871322	0.11883430					
0.11896858	0.11917761	0.11915254				
0.11918281	0.11921644	0.11926239	0.11926279			
0.11923932	0.11929081	0.11928166	0.11929784	0.11929763		
0.11927377	0.11929113	0.11929793	0.11930139	0.11930280	0.11930341	
0.11928764	0.11930307	0.11930180	0.11930348	0.11930349	0.11930364	0.11930358
0.11929536	0.11930239	0.11930317	0.11930361	0.11930364	0.11930351	
0.11929911	0.11930365	0.11930353	0.11930367	0.11930360		
0.11930118	0.11930358	0.11930364	0.11930355			
0.11930228	0.11930369	0.11930368				
0.11930291	0.11930379					
0.11930326						

TABLE IV

(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
0.00027611						
0.00034455	−0.00029420					
−0.00020326	−0.00010700	−0.00025586				
−0.00018859	−0.00028701	−0.00023467	−0.00028159			
−0.00025257	−0.00024862	−0.00027685	−0.00027000	−0.00028146		
−0.00026186	−0.00028625	−0.00027618	−0.00028161	−0.00027934	−0.00028014	
−0.00027217	−0.00027550	−0.00027961	−0.00027936	−0.00027999	−0.00027987	−0.00027992
−0.00027558	−0.00028095	−0.00027968	−0.00028007	−0.00027991	−0.00027987	
−0.00027779	−0.00027946	−0.00027991	−0.00027986	−0.00027993		
−0.00027878	−0.00028005	−0.00027989	−0.00027993			
−0.00027934	−0.00027995	−0.00028014				
−0.00027963	−0.00028006					
−0.00027981						

successively to each new series until a final limiting value is formed. This operation is illustrated in Tables III and IV, where successive columns contain the series which has been derived from the previous column. In this example, at least, these operations provide a final estimate in column (vii) that is very close to the "exact" eigenvalue. It is sometimes possible to improve the accuracy still further by repeating the above procedure on the "diagonal" elements of the arrays formed from the last value in each column which are shown underlined in Tables III and IV. The result of applying the operation to this series of diagonal elements gives finally

$$\beta_r = 0.11930374 \quad \text{and} \quad \beta_i = -0.00027995,$$

which are only in error by 2 in the eight decimal place, and provides a great improvement over the values originally estimated in Table II.

The third example (c) is also of the above type, but in this case convergence is slow with respect to m instead of n, because here $\alpha^2 - \alpha_0{}^2$ is large. The result of applying the Shanks process to the partial sums with respect to m is

$$\beta_r = 0.04021916 \quad \text{and} \quad \beta_i = 0.00278070,$$

which may be compared with the direct Orr–Sommerfeld solution

$$\beta_r = 0.04021914 \quad \text{and} \quad \beta_i = 0.00278070.$$

TABLE V

q	$S_r(q)$	$S_i(q)$
0	0.09669762	0.00483926
1	0.09063906	0.00029795
2	0.09656850	−0.00302813
3	0.09195843	−0.00014283
4	0.09545608	−0.00297762
5	0.09273801	−0.00040554
6	0.09489111	−0.00278178
7	0.09315156	−0.00060530
8	0.09457801	−0.00260579
9	0.09339249	−0.00076808
10	0.09438862	−0.00245878
11	0.09354325	−0.00090214
12	0.09426702	−0.00233695
13	0.09364219	−0.00101313
14	0.09418591	−0.00223603
Exact value	0.09393234	−0.00164860

A final example, (d), is given for a case where the original series is barely converging at all. Table V contains the first 14 terms of the series of partial sums defined by (3) for $\alpha = 0.3$ and $R = 3{,}000$. The final result obtained by repeatedly applying the transformation to this series is

$$\beta_r = 0.09393246 \quad \text{and} \quad \beta_i = -0.00164863$$

and the error lies in the seventh decimal place.

The nonlinear transformation defined by (5) applies to a one-dimensional array of numbers and before the process can be used on a double series some manipulation is needed to reduce this to an appropriate form. When the series converges very rapidly in one dimension the procedure adopted in examples (b) and (c) is appropriate, and when convergence is similar in both parameters perhaps the summation scheme that was used in (d) is a sensible choice. Another way of reducing the double summation to a single sequence is to sum along diagonals of the array so that the qth element contains all the terms in the series with $n + m$ up to q,

$$S(q) = \sum_{n=0}^{q} \sum_{m=0}^{q-n} A_{nm}(\alpha R - (\alpha R)_0)^n (\alpha^2 - \alpha_0{}^2)^m.$$

This form, which has been used to calculate all the results that are to be presented here, is suggested by the diagonal character of the coefficients shown in Fig. 1. Although this way of summing the series does not always produce as high an accuracy as can be achieved in some cases by other schemes, less coefficients are used and the process is consequently faster. For the four test cases this method gave the values

(a)	0.11930216	−0.00027987
(b)	0.06312288	0.00315665
(c)	0.04021912	0.00278071
(d)	0.09393452	−0.00164789

which are within one part in 10^{-6} of the exact values.

6. Applications

To demonstrate the power of the series approach discussed in the last section some examples are presented that require the evaluation of a large number of eigenvalues. First, amplification contours for temporal modes, β_i, are displayed in Fig. 2 with axes α and Reynolds number R. The phase velocity information for these modes is also shown. In order to obtain this contour plot it was necessary to compute eigenvalues at 2500 mesh points. These calculations took roughly the same time as that required to find 5 eigenvalues to the same accuracy directly from the Orr–Sommerfeld

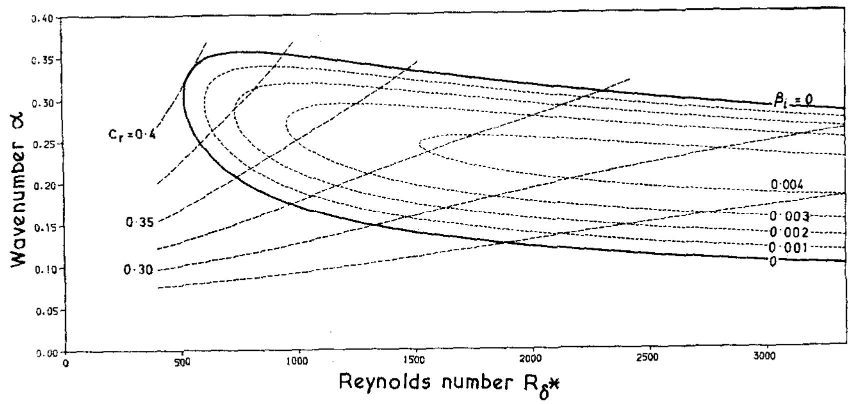


FIG. 2. Temporal growth rates.

equation by the shooting method, and indicates an improvement of some 500 to 1 in computing speed, once all the A_{nm} coefficients have been determined.

It is quite straightforward to extend the analysis to incorporate oblique waves in the method. Squire's [5] transformation provides the necessary link between an oblique mode of the form

$$\sim \exp i(ax + bz - \omega t)$$

at a Reynolds number $\hat{R}$, and the two-dimensional mode

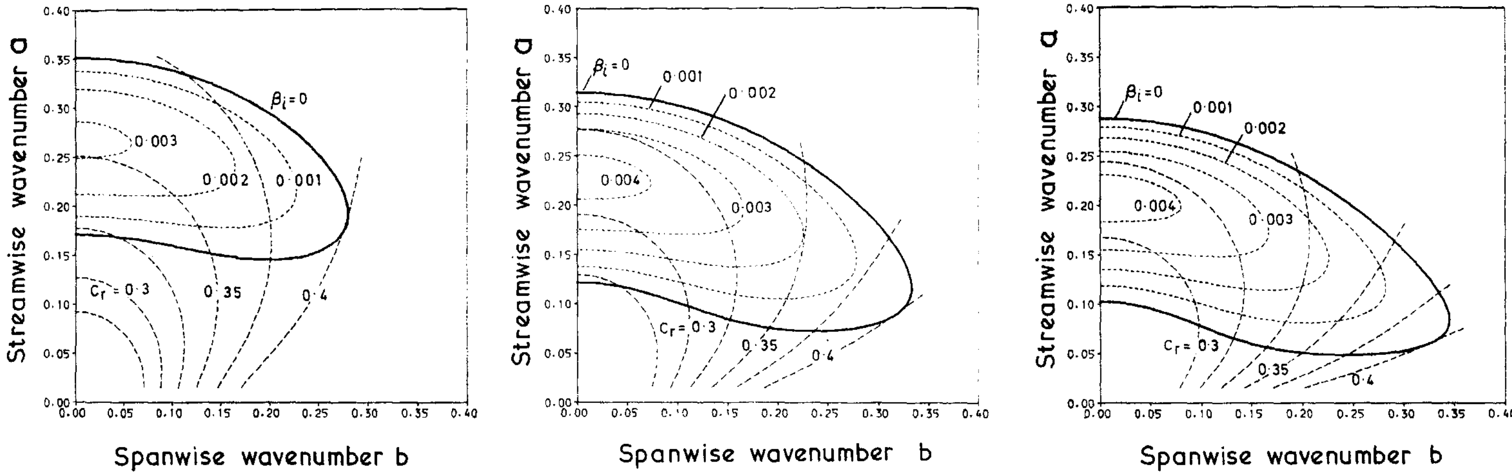
$$\exp i(\alpha x - \beta t)$$

having a Reynolds number R, through the relations

$$\alpha^2 = a^2 + b^2, \qquad \beta/\alpha = \omega/a, \qquad \text{and} \qquad \alpha R = a\hat{R}, \tag{6}$$

where a and b are the wavenumbers in the direction of the free stream and across the span, respectively. In the case of temporally growing waves, where all the wavenumbers are real, the application of this transformation is quite straightforward, R is a real quantity less than $\hat{R}$, and the growth rate of an oblique wave is given in terms of a two-dimensional wave at this lower Reynolds number. Waves growing spatially in the free-stream direction having real values of ω and b, or more general modes with exponential behaviour in both space and time, also relate to two-dimensional ones, but with a complex 'Reynolds number' parameter R. The series expressions that have been developed for the calculation of two-dimensional eigenvalues are, in fact, valid for complex values of both α and R, and there is no difficulty in applying the series description to three-dimensional modes. The series may be written in the form

$$\frac{\omega}{a} = \sum\sum A_{nm}(aR - (\alpha R)_0)^n(a^2 + b^2 - \alpha_0^{\,2})^m.$$



(i) Reynolds number 1000 (ii) Reynolds number 2000 (iii) Reynolds number 3000

FIG. 3. Temporal amplification of oblique wave.

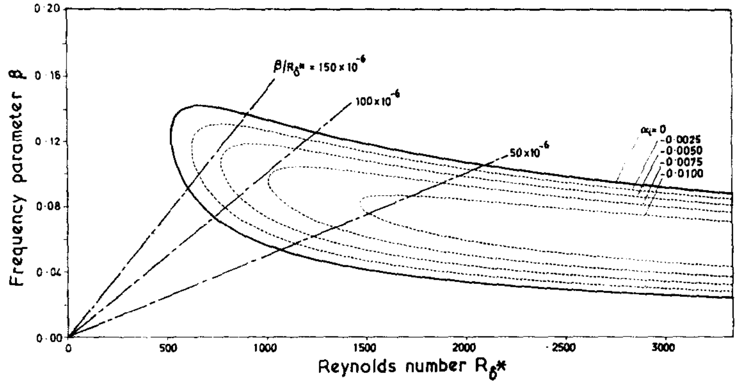


FIG. 4. Spatial amplification rates.

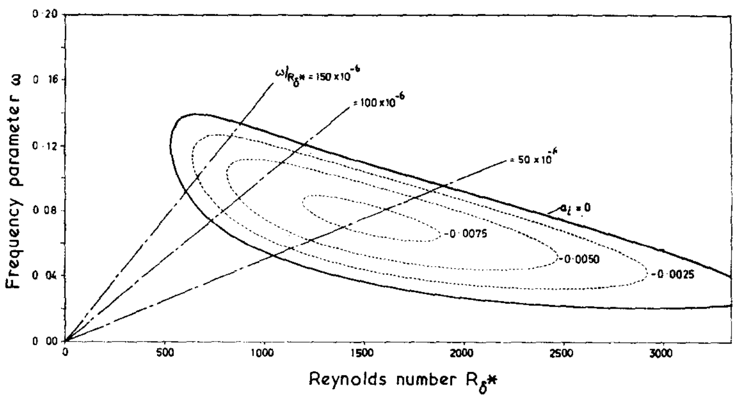


FIG. 5. Spatial amplification of oblique waves when $b = R_\delta*/10000$.

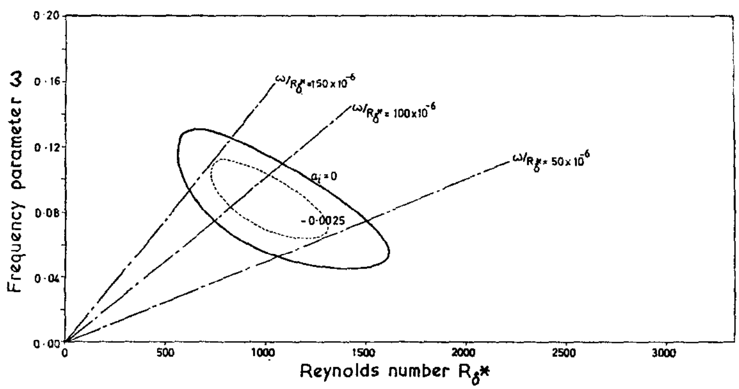


FIG. 6. Spatial amplification when $b = R_\delta*/5000$.

Figure 3 shows a set of "kidney" curves for the temporal amplification rate, β_i, plotted on an $a \sim b$ plane for three values of the Reynolds number evaluated from the above series.

Spatial growth in the stream direction is described by modes with a complex wavenumber, a, and real values of ω and b. These modes can be obtained from the series representation of ω by using an iteration scheme arranged so that the wavenumber is found for a specified real ω, b, and Reynolds number R. Figure 4, which shows these spatial amplification contours for two-dimensional modes ($b = 0$) also contains 2500 grid points. The paths along which the real frequency $\omega/R_\delta{}^*$ remains constant are shown as chain-dotted lines. Similar information for oblique spatially growing modes has also been calculated from the series for spanwise wavenumbers equal to $b/R_\delta{}^* = 10{,}000$ and $b/R_\delta{}^* = 5000$, and these are shown in Figs. 5 and 6.

7. Discussion and Concluding Remarks

A series representation of the eigenvalue relationship for the locally parallel model of the Blasius boudary layer profile has been developed. It has been shown how eigenvalues can be determined to high accuracy from relatively few terms of this series with the aid of a nonlinear transformation to make the series more convergent. Although no new results have been obtained in this exercise the power and speed of the technique have been amply demonstrated.

Slowly convergent series can often be rendered amenable to summation by a change of origin, and no doubt a suitable set of series expansions could have been derived in the present case to cover different regions of the α^2 and αR planes. In fact the transformation used here effectively removes the divergent behavior that arises from an inappropriate choice of origin. The radius of convergence of a series is defined as the ratio of the nth to the $(n-1)$th term as n approaches infinity. In cases where the coefficients of the series are evaluated on a computer, rounding errors will eventually limit the series to a finite number of terms, and estimates of the radius of convergence have to be made from this finite sequence. Nevertheless, quite often the ratio of the magnitudes of successive terms seems to reach a well-defined limit after relatively few terms. An example of this behavior is provided by the A_{nm} coefficients shown in Fig. 1 which exhibit a systematic decaying pattern for n and m above about 10. When a well-defined character is apparent in a series it is clearly efficient computationally to sum the tail of the series by analytical means. The nonlinear transformation that has been used here does in fact sum geometric series exactly; it will even give the "anti-limit" of an exponentially divergent series. Repeated application of the transformation filters out those components of the series that exhibit well-defined exponential or oscillatory behavior. The choice of origin should thus have little effect on the final result, although in practice rounding errors make it advisable to select an origin such that reasonably convergent series occur. The Shanks transformation is related to the Padé approximant which also has this invariant property.

Van Dyke [6] presents a number of examples where techniques for improving the convergence of a series are used to apparently "predict" the behavior of some function well outside the scope of the analysis used to form the original series. Nothing as ambitious is attempted here, all the results that are obtained can all be checked against direct solutions of the differential equation, and in no sense is the technique used to provide information that cannot be computed directly. It does not seem to be necessary to understand precisely how the transformation achieves this end; for our purpose it is sufficient that it does indeed improve the convergence of the series defining eigenvalues of the Orr–Sommerfeld equation over the range of parameters in which we are interested. Although there have only been a few isolated comparisons made between eigenvalues from the Shanks' limit and those obtained directly from the differential equation, it seems that over the parameter space of interest in the instability problem estimates can be made to an accuracy of at least one part in 10^{-6} by applying the simple nonlinear transformation to the series formed from the diagonal summations. For all practical purposes this degree of accuracy is quite acceptable.

The Shanks process effectively finds a pattern in the way the series develops and then evaluates the sum of the extrapolated sequence. To succeed it is essential that the truncated series used for this be very accurate. The contouring method of calculating the coefficients of the expansion produces a finite series that exactly reflects the data points on the contouring circle, and the terms of highest degree may not be precisely equal to the truncated power series. This occurs when insufficient points are used to represent the behavior of the eigenvalues in the chosen region, and the coefficients of the series become aliased by the remainder of the series that has necessarily been neglected. A small radius helps to overcome this problem, but difficulties can then arise through rounding errors which introduce other inaccuracies in the higher coefficients. Radii have been chosen here to provide a compromise between these conflicting requirements.

Since all the circular contours in the α^2 and αR planes map onto closed simply connected circuits in the β plane, it is clear that there are no branch points within the chosen domain. There are, nevertheless, branch points outside the domain. In fact, originally somewhat larger circuits were chosen for the contouring procedure, and difficulty was experienced when the shooting process sometimes converged onto higher modes which had eigenvalues similar to those of the lowest mode. The series however it is summed, can only reflect the behavior of the eigenvalues from which the coefficients were formed. Attempts to extend the series evaluation into regions close to these branch points, where the eigenvalue relations cease to be analytic, are therefore likley to lead to erroneous results. The fact that Shanks' transformation applied to the series renders it convergent in some of these cases does not imply that the resulting limit is a correct one. Incorrect limits occur in particular when the series is used to predict eigenvalues at high Reynolds numbers, where it can be expected that there will be a significant number of possible higher modes present [7]. The series in the form chosen will always be analytic, and it cannot represent the true eigenvalue behavior in those regions where the function is multivalued. Presumably by using a more appropriate expansion containing the necessary singular behavior, in the manner of [3], the range of validity could be extended to cope with these situations.

The nonlinear transformation that has been used to improve the convergence of the series representation of the eigenvalue relations is related to the Padé approximant, and it seems very likely that representation by these rational fractions offers a useful alternative scheme. Such an approach may well be computationally faster as well as being more convenient to implement. The formation of the expansion of the characteristic function in terms of Padé approximants in the three variables $(\alpha^2, \alpha R, \beta/\alpha)$ also offers the prespect of propertly identifying the zeros of the function and thus of obtaining a complete description of the higher modes.

The series method can almost certainly be applied to other velocity profile shapes, and once the coefficients for a suitable family of profiles have been evaluated the calculation of the development of an instability wave around an aerofoil, say, can be reduced to a relatively simple set of operations. It is expected that the influence of boundary layer growth could also be incorporated in such an extension of the technique, but so far this has not been attempted. The objective of this paper was to show that in cases where large numbers of eigenvalues need to be evaluated, it may be well worth considering the use of series expansions.

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